# Chapter 13

# Grappling with Graphing

### In This Chapter

- Looking at lines and their special features
- Investigating intersections of lines
- Taking a peek at parabolas

In this chapter, I present the basics for working with lines and their equations. You find lines determined by two points and then other lines determined by a slope and a point. You see lines that meet and lines that avoid one another forever.

I also throw you a curve or two! Circles and parabolas are the most recognizable of the algebraic curves and have the most respectable equations. The basics for drawing these curves is found here.

## Preparing to Graph a Line

A straight *line* is the set of all the points on a graph that satisfy a linear equation. When any two points on a line are chosen, the *slope* of the segment between those two points is always the same number.

To graph a line, you need only two points. A rule in geometry says that only one line can go through two particular points. Even though only two points are needed to graph a line, it's usually a good idea to graph at least three points to be sure that you graphed the line correctly.

An equation whose graph is a straight line is said to be *linear.* A linear equation has a standard form of $ax + by = c$, where $x$ and $y$ are variables and $a$, $b$, and $c$ are real numbers. A point

$(x, y)$ lies on the line if the $x$ and $y$ make the equation true. When graphing a line, you can find some pairs of numbers that make the equation true and then connect them. Connect the dots!

Graphing lines from their equations just takes finding enough points on the line to convince you that you've drawn the graph correctly.

Find a point on the line $x - y = 3$.

1. **Choose a random value for one of the variables, either $x$ or $y$.**

   To make the arithmetic easy for yourself, pick a large enough number so that, when you subtract $y$ from that number, you get a positive 3. In $x - y = 3$, you can let $x = 8$, so $8 - y = 3$.

2. **Solve for the value of the other variable.**

   Subtract 8 from each side to get $-y = -5$.

   Multiply each side by $-1$ to get $y = 5$.

3. **Write an ordered pair for the coordinates of the point.**

   You chose 8 for $x$ and solved to get $y = 5$, so your first ordered pair is (8, 5).

You can find more ordered pairs by choosing another number to substitute for either $x$ or $y$.

Find a point that lies on the line $2x + 3y = 12$.

1. **Solve the equation for one of the variables.**

   Solving for $y$ in the sample problem, $2x + 3y = 12$, you get

   $$3y = 12 - 2x$$

   $$y = \frac{12 - 2x}{3}$$

   With multipliers involved, you often get a fraction.

2. **Choose a value for the other variable and solve the equation.**

   Try to pick values so that the result in the numerator is divisible by the 3 in the denominator — giving you an integer.

   For example, let $x = 3$. Solving the equation:

   $$y = \frac{12 - 2 \cdot 3}{3} = \frac{6}{3} = 2$$

   So, the point (3, 2) lies on the line.

# Incorporating Intercepts

An *intercept* of a line is a point where the line crosses an axis. Unless a line is vertical or horizontal, it crosses both the $x$ and $y$ axes, so it has two intercepts: an $x$-intercept and a $y$-intercept. Horizontal lines have just a $y$-intercept, and vertical lines have just an $x$-intercept. The exceptions are when the horizontal line is actually the $x$-axis or the vertical line is the $y$-axis. Intercepts are quick and easy to find and can be a big help when graphing.

The $x$-intercept of a line is where the line crosses the $x$-axis. To find the $x$-intercept, let the $y$ in the equation equal 0 and solve for $x$.

Find the $x$-intercept of the line $4x - 7y = 8$.

First, let $y = 0$ in the equation. Then:

$$\begin{aligned} 4x - 0 &= 8 \\ 4x &= 8 \\ x &= 2 \end{aligned}$$

The $x$-intercept of the line is (2, 0). The line goes through the $x$-axis at that point.

The $y$-intercept of a line is where the line crosses the $y$-axis. To find the $y$-intercept, let the $x$ in the equation equal 0 and solve for $y$.

Find the $y$-intercept of the line $3x - 7y = 28$.

First, let $x = 0$ in the equation. Then:

$$0 - 7y = 28$$
$$-7y = 28$$
$$y = -4$$

The $y$-intercept of the line is (0, –4).

As long as you're careful when graphing the $x$- and $y$-intercepts and get them on the correct axes, the intercepts are often all you need to graph a line.

# Sliding the Slippery Slope

The slope of a line is a number that describes the steepness and direction of the graph of the line. The slope is a positive number if the line moves upward from left to right; the slope is a negative number if the line moves downward from left to right. The steeper the line, the greater the absolute value of the slope (the farther the number is from 0).

Knowing the slope of a line beforehand helps you graph the line. You can find a point on the line and then use the slope and that point to graph it. A line with a slope of 6 goes up steeply. If you know what the line should look like (that is, whether it should go up or down) — information you get from the slope — you'll have an easier time graphing it correctly.

Figure 13-1 shows some lines with their slopes. The lines are all going through the origin just for convenience.

What about a horizontal line — one that doesn't go upward or downward? A horizontal line has a 0 slope. A vertical line has no slope; the slope of a vertical line (it's so steep) is undefined.

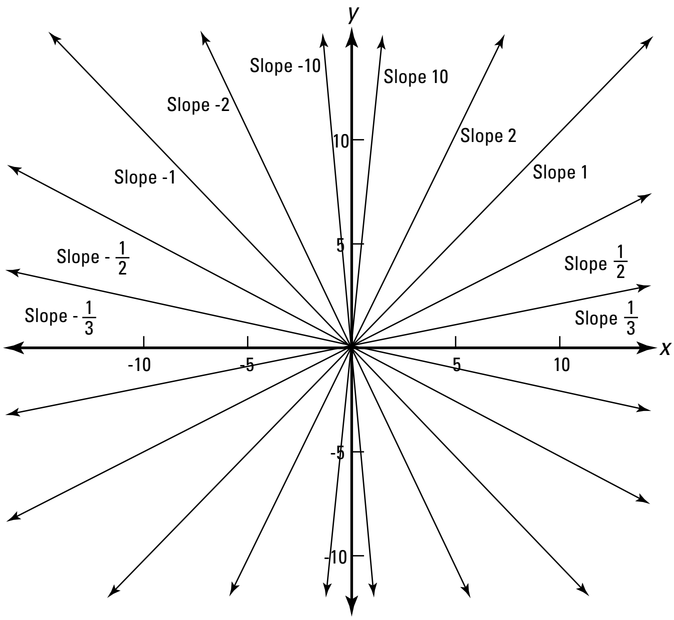



**Figure 13-1:** Pick a line — see its slope.

TIP

One way of referring to the slope, when it's written as a fraction, is rise over run. If the slope is $\frac{3}{2}$, it means that for every 2 units the line runs left to right along the *x*-axis, it rises 3 units along the *y*-axis. A slope of $\frac{-1}{8}$ indicates that as the line runs 8 units horizontally, parallel to the *x*-axis left to right, it drops (negative rise) 1 unit vertically.

## Computing slope

If you know two points on a line, you can compute the number representing the slope of the line.

ALGEBRA RULES

The slope of a line, denoted by the small letter *m*, is found when you know the coordinates of two points on the line, $(x_1, y_1)$ and $(x_2, y_2)$:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Subscripts are used here to identify which is the first point and which is the second point. There's no rule as to which is which; you can name the points any way you want. It's just a good idea to identify them to keep things in order. Reversing the points in the formula gives you the same slope (when you subtract in the opposite order):

$$m = \frac{y_1 - y_2}{x_1 - x_2}$$

You just can't mix them and do $(y_1 - y_2)$ over $(x_2 - x_1)$.

Now, you can see how to compute slope with the following examples.

Find the slope of the line going through (3, 4) and (2, 10).

Let (3, 4) be $(x_1, y_1)$ and (2, 10) be $(x_2, y_2)$. Substitute into the formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{10 - 4}{2 - 3}$$

Simplify:

$$m = \frac{6}{-1} = -6$$

This line is pretty steep as it falls from left to right.

Find the slope of the line going through (4, 2) and (–6, 2).

Let (4, 2) be $(x_1, y_1)$ and (–6, 2) be $(x_2, y_2)$. Substitute into the formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 2}{-6 - 4}$$

Simplify:

$$m = \frac{0}{-10} = 0$$

These points are both 2 units above the *x*-axis and determine a horizontal line. That's why the slope is 0.

Find the slope of the line going through (2, 4) and (2, –6).

Let (2, 4) be $(x_1, y_1)$ and (2, –6) be $(x_2, y_2)$. Substitute into the formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-6 - 4}{2 - 2}$$

Simplify:

$$m = \frac{-10}{0}$$

Oops! You can't divide by 0. There is no such number. The slope doesn't exist or is undefined. These two points are on a vertical line.

Watch out for these common errors when working with the slope formula:

- **Be sure that you subtract the *y* values on the top of the division formula.** A common error is to subtract the *x* values on the top.
- **Be sure to keep the numbers in the same order when you subtract.** Decide which point is first and which point is second. Then take the second *y* minus the first *y* and the second *x* minus the first *x*. Don't do the top subtraction in a different order from the bottom.

## Combining slope and intercept

An equation of a single line can take many forms. Just as you can solve for one variable or another in a formula, you can solve for one of the variables in the equation of a line. This change of format can help you find the points to graph the line or find the slope of a line.

A common and popular form of the equation of a line is the *slope-intercept form.* It's given this name because the slope of the line and the *y*-intercept of the line are obvious on sight. When a line is written $6x + 3y = 5$, you can find points by plugging in numbers for *x* or *y* and solving for the other coordinate. But, by using methods for solving linear equations (see Chapter 6), the same equation can be written $y = -2x + \frac{5}{3}$, which tells you that the slope is –2 and the place where the line crosses the $y$-axis (the $y$-intercept) is $\left(0, \frac{5}{3}\right)$.

Where $y$ and $x$ represent coordinates of a point on the line, $m$ is the slope of the line, and $b$ is the $y$-intercept of the line, the slope-intercept form is $y = mx + b$.

In every case shown next, the equation is written in the slope-intercept form. The coefficient of $x$ is the slope of the line and the constant gives the $y$-intercept.

- $\boldsymbol{y = 2x + 3}$**:** The slope is 2; the $y$-intercept is (0, 3).
- $\boldsymbol{y = \frac{1}{3}x - 2}$**:** The slope is $\frac{1}{3}$; the $y$-intercept is (0, –2).
- $\boldsymbol{y = 7}$**:** The slope is 0; the $y$-intercept is (0, 7). You can read this equation as being $y = 0 \cdot x + 7$.

## Creating the slope-intercept form

If the equation of the line isn't already in the slope-intercept form, solving for $y$ changes the equation to slope-intercept form.

Put the equation $5x - 2y = 10$ in slope-intercept form.

1. **Get the $y$ term by itself on the left.**

   Subtract $5x$ from each side to get the $y$ term alone:

   $$-2y = -5x + 10$$

2. **Solve for $y$.**

   Divide each side by –2 and simplify the two terms on the right:

   $$\frac{-2y}{-2} = \frac{(-5x + 10)}{-2}$$
   $$y = \frac{-5x}{-2} + \frac{10}{-2}$$
   $$y = \frac{5}{2}x - 5$$

   The slope is $\frac{5}{2}$ and the $y$-intercept is at (0, –5).

## Graphing with slope-intercept

One advantage to having an equation in the slope-intercept form is that graphing the line can be a fairly quick task, as the following example shows.

EXAMPLE

Graph $y = \frac{3}{2}x + 1$.

The slope of this line is $\frac{3}{2}$ and the $y$-intercept is the point (0, 1). First, graph the $y$-intercept (see Figure 13-2). Then use the rise-over-run interpretation of slope to count spaces to another point on the line. To do this, do the run, or bottom, movement first. In this sketch, move 2 units to the right of (0, 1). From there, rise (or go up) 3 units, which should get you to (2, 4).

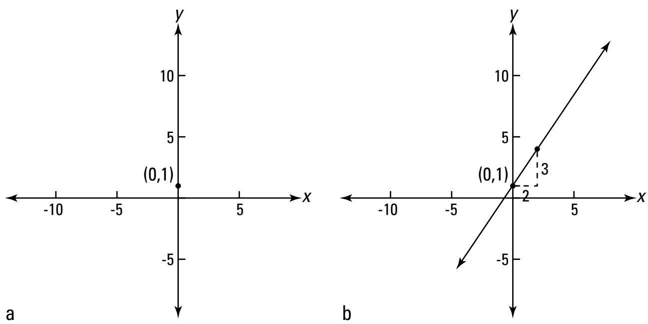



**Figure 13-2:** The $y$-intercept is located; use run and rise to find another point.

It's sort of like going on a treasure hunt: "Two steps to the east; three steps to the north; now dig in!" Only our "dig in" is to put a point there and connect that point with the starting point — the intercept. Look at the right-hand side (the b side) of Figure 13-2 to see how it's done.

# Making Parallel and Perpendicular Lines Toe the Line

The slope of a line gives you information about a particular characteristic of the line. It tells you if it's steep or flat and if it's rising or falling as you read from left to right. The slope of a line can also tell you if one line is parallel or perpendicular to another line.

Parallel lines never touch. They're always the same distance apart and never share a common point. They have the same slope.

Perpendicular lines form a 90-degree angle (a *right angle*) where they cross. They have slopes that are negative reciprocals of one another. For example, the $x$-axis and $y$-axis are perpendicular lines.

If line $l_1$ has a slope of $m_1$, and if line $l_2$ has a slope of $m_2$, then the lines are parallel if $m_1 = m_2$. If line $l_1$ has a slope of $m_1$, and if line $l_2$ has a slope of $m_2$, then the lines are perpendicular if $m_1 = -\frac{1}{m_2}$ or if they are horizontal or vertical lines.

The following examples show you how to determine whether lines are parallel or perpendicular by just looking at their slopes:

- The line $y = 3x + 2$ is parallel to the line $y = 3x - 7$ because their slopes are both 3.
- The line $3x + 2y = 8$ is parallel to the line $6x + 4y = 7$ because their slopes are both $\frac{-3}{2}$. Write each line in the slope-intercept form to see this: $3x + 2y = 8$ can be written $y = \frac{-3}{2}x + 4$ and $6x + 4y = 7$ can be written $y = \frac{-3}{2}x + \frac{7}{4}$.
- The line $y = \frac{3}{4}x + 5$ is perpendicular to the line $y = \frac{-4}{3}x + 6$ because their slopes are negative reciprocals of one another.
- The line $y = -3x + 4$ is perpendicular to the line $y = \frac{1}{3}x - 8$ because their slopes are negative reciprocals of one another.

# Criss-Crossing Lines

If two lines *intersect,* or cross one another, then they intersect exactly once and only once. The place they cross is the point of intersection and that common point is the only one both lines share. Careful graphing can sometimes help you to find the point of intersection.

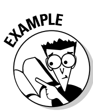
The point (5, 1) is the point of intersection of the two lines $x + y = 6$ and $2x - y = 9$ because the coordinates make each equation true:

- If $x + y = 6$, then substituting the values $x = 5$ and $y = 1$ give you $5 + 1 = 6$, which is true.
- If $2x - y = 9$, then substituting the values $x = 5$ and $y = 1$ give $2 \cdot 5 - 1 = 10 - 1 = 9$, which is also true.

This is the only point that works for both the lines.

One way to find the intersection of two lines is to graph both lines (very carefully) and observe where they cross. This technique is not very helpful when the intersection has fractional coordinates, though.

Another way to find the point where two lines intersect is to use a technique called *substitution* — you substitute the $y$ value from one equation for the $y$ value in the other equation and then solve for $x$. Because you're looking for the place where $x$ and $y$ of each line are the same — that's where they intersect — then you can write the equation $y = y$, meaning that the $y$ from the first line is equal to the $y$ from the second line. Replace the $y$'s with what they're equal to in each equation, and solve for the value of $x$ that works.




Find the intersection of the lines $3x - y = 5$ and $x + y = -1$.

1. **Put each equation in the slope-intercept form, which is a way of solving each equation for *y*.**

   $3x - y = 5$ is written as $y = 3x - 5$, and $x + y = -1$ is written as $y = -x - 1$. (The lines are not parallel, and their slopes are different, so there will be a point of intersection.)

2. **Set the *y* points equal and solve.**

   From $y = 3x - 5$ and $y = -x - 1$, you substitute what $y$ is equal to in the first equation with the $y$ in the second equation: $3x - 5 = -x - 1$.

**3. Solve for the value of *x*.**

Add $x$ to each side and add 5 to each side:

$$3x + x - 5 + 5 = -x + x - 1 + 5$$
$$4x = 4$$
$$x = 1$$

Substitute that 1 for $x$ into either equation to find that $y = -2$. The lines intersect at the point (1, –2).

# Turning the Curve with Curves

A circle is a most recognizable shape. A circle is basically all the points that are a set distance from the point called the circle's *center.* A parabola isn't quite as recognizable as a circle, but it's represented by a quadratic equation and fairly easy to graph.

## Going around in circles with a circular graph

An example of an equation of a circle is $x^2 + y^2 = 25$. The circle representing this equation goes through an infinite number of points. Here are just some of those points:

| | | | | |
|---|---|---|---|---|
| (0, 5) | (0, –5) | (5, 0) | (–5, 0) | (3, 4) |
| (4, 3) | (4, –3) | (–3, 4) | (–3, –4) | (–4, –3) |

I haven't finished all the possible points with integer coordinates, let alone points with fractional coordinates, such as $\left(\frac{25}{13}, \frac{60}{13}\right)$.

When graphing an equation, you don't expect to find all the points. You just want to find enough points to help you sketch in all the others without naming them.

## Putting up with parabolas

Parabolas are nice, U-shaped curves. They're the graphs of quadratic equations where either an $x$ term is squared or a $y$ term is squared, but not both are squared at the same time. Parabolas have a highest point or a lowest point (or the farthest left point or the farthest right point) called the *vertex.*

### Trying out the basic parabola

My favorite example of a parabola is $y = x^2$, the basic parabola. Figure 13-3 shows a graph of this formula. This equation says that the $y$-coordinate of every point on the parabola is the square of the $x$-coordinate.

The vertex of the parabola in Figure 13-3 is at the origin, (0, 0), and the graph curves upward.

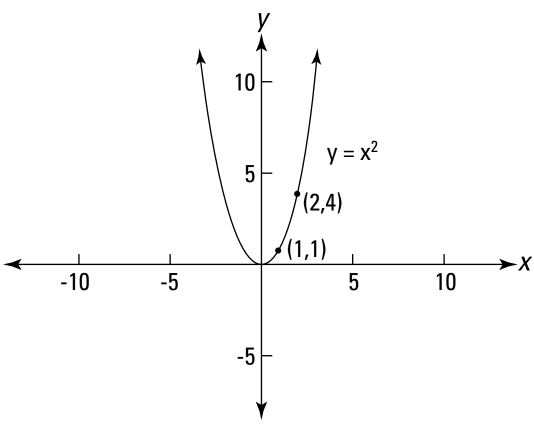



**Figure 13-3:** The simplest parabola.

You can make this parabola steeper or flatter by multiplying the $x^2$ by certain numbers. If you multiply the squared term by numbers bigger than 1, it makes the parabola steeper. If you multiply by numbers between 0 and 1 (which are proper fractions), it makes the parabola flatter.

You can make the parabola open downward by multiplying the $x^2$ by a negative number, and make it steeper or flatter than the basic parabola — in a downward direction.

### Putting the vertex on an axis

The basic parabola, $y = x^2$, can be slid around — left, right, up, down — placing the vertex somewhere else on an axis and not changing the general shape.

If you change the basic equation by adding a constant number to the $x^2$ — such as $y = x^2 + 3$, $y = x^2 + 8$, $y = x^2 - 5$, or $y = x^2 - 1$ — then the parabola moves up and down the $y$-axis. Note that adding a negative number is also part of this rule. These manipulations help make a parabola fit the model of a certain situation.

If you change the basic parabolic equation by adding a number to the $x$ first and then squaring the expression — such as $y = (x + 3)^2$, $y = (x + 8)^2$, $y = (x - 5)^2$, or $y = (x - 1)^2$ — you move the graph to the left or right of where the basic parabola lies. Using +3, as in the equation $y = (x + 3)^2$, moves the graph to the left, and using –3, as in the equation $y = (x - 3)^2$, moves the graph to the right. It's the opposite of what you might expect, but it works this way consistently.

EXAMPLE

The following equations and their graphs are shown in Figure 13-4:

- $\boldsymbol{y = 3x^2 - 2}$**:** The 3 multiplying the $x^2$ makes the parabola steeper, and the –2 moves the vertex down to (0, –2).
- $\boldsymbol{y = \frac{1}{4}x^2 + 1}$**:** The $\frac{1}{4}$ multiplying the $x^2$ makes the parabola flatter, and the +1 moves the vertex up to (0, 1).
- $\boldsymbol{y = -5x^2 + 3}$**:** The –5 multiplying the $x^2$ makes the parabola steeper and causes it to go downward, and the +3 moves the vertex to (0, 3).
- $\boldsymbol{y = 2(x - 1)^2}$**:** The 2 multiplier makes the parabola steeper, and subtracting 1 moves the vertex right to (1, 0).

- $y=-\frac{1}{3}(x+4)^2$**:** The $-\frac{1}{3}$ makes the parabola flatter and causes it to go downward, and adding 4 moves the vertex left to (–4, 0).
- $y=-\frac{1}{20}x^2+5$**:** The $-\frac{1}{20}$ multiplying the $x^2$ makes the parabola flatter and causes it to go downward, and the +5 moves the vertex to (0, 5).

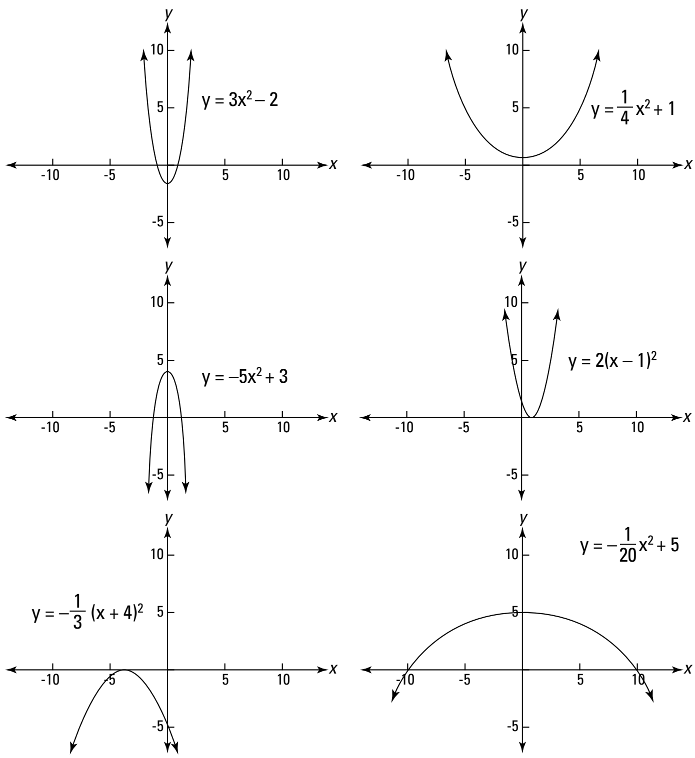



**Figure 13-4:** Parabolas galore.